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Hilbert–Kunz Multiplicity and an Inequality between Multiplicity and Colength

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In this paper, we study local rings of small Hilbert–Kunz multiplicity. In particular, we prove that an unmixed local ring of Hilbert–Kunz multiplicity one is regular and classify two-dimensional Cohen–Macaulay local rings whose Hilbert–Kunz multiplicity is 2 or less. Also, we investigate the inequality between the multiplicity and the colength of the tight closure of parameter ideals inverse to the usual inequality between multiplicity and colength.

Key Words: Hilbert–Kunz multiplicity; regular local rings; Cohen–Macaulay local rings; rational singularity; Frobenius endomorphism; tight closure

1. INTRODUCTION

Throughout this paper, let $(A, \mathfrak{m})$ be a commutative Noetherian local ring of dimension $d := \dim A$ with unique maximal ideal $\mathfrak{m}$ and the residue class field $k = A/\mathfrak{m}$. Further, let $\widehat{A}$ denote the $\mathfrak{m}$-adic completion of A. For a finite A-module M, we denote by $\mathrm{Ass}_A(M)$(resp. $\mathrm{Min}_A(M)$) the associated prime ideals (resp. the minimal prime ideals) of M. Moreover,

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we put

$$\mathrm{Assh}_A(M) = \{P \in \mathrm{Ass}_A(M) \mid \dim A/P = \dim M\}.$$

A local ring A is *unmixed* (resp. *quasi-unmixed*) if $\mathrm{Ass}(\widehat{A}) = \mathrm{Assh}(\widehat{A})$ (resp. $\mathrm{Min}(\widehat{A}) = \mathrm{Assh}(\widehat{A})$).

First, we recall the definition of usual multiplicity.

DEFINITION 1.1. Let I be an $\mathfrak{m}$-primary ideal and let M be a finite A-module. The *multiplicity* of M with respect to I is defined as

$$e(I, M) := \lim_{n\to\infty} \frac{l_A(M/I^nM)\cdot d!}{n^d}. \tag{1.1.1}$$

By definition the multiplicity of A is $e(A) = e(\mathfrak{m}, A)$.

Kunz [Ku1] proved the following theorem, which gives a characterization of regular local rings of characteristic $p > 0$.

KUNZ' THEOREM [Ku1]. *Let A be a local ring of characteristic $p > 0$. Then the following conditions are equivalent.*

(1) *A is a regular local ring.*

(2) *A is reduced and it is flat over $A^p = (a^p \mid a \in A)$.*

(3) *$l_A(A/\mathfrak{m}^{[q]}) = q^d$ for all $q = p^e$, $e \geq 1$, where $\mathfrak{m}^{[q]} = (a^q \mid a \in \mathfrak{m})A$.*

Furthermore, Kunz [Ku1] proved the following inequality for any local ring A:

$$l_A(A/\mathfrak{m}^{[q]}) \geq q^d \qquad \text{for all } q = p^e,\ e \geq 1. \tag{1.2}$$

Also, in [Ku2], Kunz observed that the number $\lambda_e(A) = l_A(A/\mathfrak{m}^{[q]})/q^d$ is a reasonable measure for the singularity of the local ring A and discussed its behavior under localization and flat ring extension. So, the idea of the Hilbert–Kunz multiplicity $\lim_{e\to\infty} \lambda_e(A)$ is due to Kunz and the systematic treatment was achieved by Monsky [Mo].

DEFINITION 1.3 [Mo]. Suppose that A has prime characteristic $p > 0$. Then the *Hilbert–Kunz multiplicity* of M with respect to I is defined as

$$e_{\mathrm{HK}}(I, M) := \lim_{e\to\infty} \frac{l_A(M/I^{[p^e]}M)}{p^{de}}, \tag{1.3.1}$$

where $I^{[q]}(q = p^e)$ is the ideal generated by the qth powers of the elements of I. Notice that the limit of the right-hand side of (1.3.1) always exists and is a real number; see, e.g., Monsky [Mo] or Huneke [Hu]. By definition the Hilbert–Kunz multiplicity of A is $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(\mathfrak{m}, A)$. Further, we frequently write $e_{\mathrm{HK}}(I)$ instead of $e_{\mathrm{HK}}(I, A)$.

Remark. Recently, Seibert [Se] proved that if A is a Cohen–Macaulay local ring of finite Cohen–Macaulay representation type then $e_{\mathrm{HK}}(I, A)$ is always a rational number for any $\mathfrak{m}$-primary ideal I. However, in general, it remains open whether $e_{\mathrm{HK}}(I, A)$ is always a rational number or not.

It is well known that any regular local ring has multiplicity one and the converse was proved by Nagata [Na, (40.6)]. Namely, any unmixed local ring with multiplicity one is regular.

Now suppose that a local ring A has prime characteristic $p > 0$. Then it is easy to see that a regular local ring has Hilbert–Kunz multiplicity one. More precisely, we have

1.4. If A is regular, then $e_{\mathrm{HK}}(I, A) = l_A(A/I)$ for every $\mathfrak{m}$-primary ideal I of A.

Thus it is natural to ask

QUESTION *If* $e_{\mathrm{HK}}(A) = 1$, *is* A *then regular?*

To the above question, the following theorem gives a positive answer, which is one of the main results in this paper.

THEOREM 1.5. *Let* A *be a local ring of characteristic* $p > 0$. *If* A *is unmixed with* $e_{\mathrm{HK}}(A) = 1$, *then it is regular.*

Remark. The above theorem has been proved by Han and Monsky [HM, (5.8),(5.9)] in the case of complete intersections.

In general, for any parameter ideal $\mathfrak{q}$ of a local ring A, we have $e(\mathfrak{q}, A) \le l_A(A/\mathfrak{q})$. Moreover, equality holds if and only if A is Cohen–Macaulay. In the process of discussing the above theorem, we obtained the next result as a starting point (see Section 2 about the definition of tight closures).

PROPOSITION 4.1. *Suppose* $e_{\mathrm{HK}}(A) = 1$. *Then* $e(\mathfrak{q}, A) \le l_A(A/\mathfrak{q}^*)$ *for every parameter ideal of* A, *where* $\mathfrak{q}^*$ *denotes the tight closure of* $\mathfrak{q}$.

Thus we come to the following conjecture.

Conjecture 1.6. Let A be an unmixed local ring of characteristic $p > 0$. Then

(1) For every parameter ideal $\mathfrak{q}$ of A, we have $e(\mathfrak{q}, A) \ge l_A(A/\mathfrak{q}^*)$.

(2) If $e(\mathfrak{q}, A) = l_A(A/\mathfrak{q}^*)$ holds for some parameter ideal $\mathfrak{q}$ of A, then A is Cohen–Macaulay and F-rational.

To investigate this conjecture, we use the following ideal for any system of parameters $\underline{a} = a_1, \ldots, a_d$ of A:

$$\Sigma(\underline{a}) := \sum_{i=1}^{d} (a_1, \ldots, \widehat{a_i}, \ldots, a_d) : a_i + (a_1, \ldots, a_d)A.$$

This ideal has been defined by Yamagishi and Goto in the theory of Buchsbaum modules and unconditioned strong d-sequences; see, e.g., [Go, GY].

The "colon capturing property for tight closures" yields the inequalities

$$(\underline{a}) \subseteq \Sigma(\underline{a}) \subseteq (\underline{a})^*.$$

Thus in order to obtain the required inequality, it is enough to show

$$e(\underline{a}, A) \geq l_A(A/\Sigma(\underline{a})).$$

This inequality is not true for an arbitrary system of parameters. But we can show that it holds under a mild condition (#) on the system of parameters (see (3.2) for the condition (#)). Consequently, we get the following result, which gives a partial answer to the above conjecture.

THEOREM 3.4. *Let A be an unmixed local ring which is a homomorphic image of a Gorenstein local ring. Then the following statements hold.*

(1) *For every system of parameters $\underline{a}$ of A with (#), we have $e(\underline{a}, A) \geq l_A(A/(\underline{a})^*)$.*

(2) *Further, if $e(\underline{a}, A) = l_A(A/(\underline{a})^*)$ for every system of parameters $\underline{a}$ with (#), then A is Cohen–Macaulay and F-rational.*

Remark. We have no examples of a non-Cohen–Macaulay unmixed local ring A and a system of parameters $\underline{a}$ with $e(\underline{a}, A) = l_A(A/(\underline{a})^*)$.

It follows from Theorem 3.4 and Proposition 4.1 that an unmixed local ring with Hilbert–Kunz multiplicity one is Cohen–Macaulay. In Section 4, we complete the proof of Theorem 1.5 using the above fact and Kunz' theorem.

In Section 5, we classify two-dimensional Cohen–Macaulay local rings whose Hilbert–Kunz multiplicity is 2 or less. Namely, we have

THEOREM 5.4. *Let A be Cohen–Macaulay with* $\dim A = 2$. *Then*

(1) $1 < e_{\mathrm{HK}}(A) < 2$ *if and only if A is an F-rational double point. In this case, $e_{\mathrm{HK}}(A) = 2 - \frac{1}{|G|}$, where G is the finite subgroup of $SL(2, k)$ attached to the corresponding singularity in characteristic* 0.

(2) $e_{\mathrm{HK}}(A) = 2$ *if and only if A is either a non-F-rational double point or the "ordinary triple point"* $(\widehat{A} \cong k[[x^3, x^2y, xy^2, y^3]])$.

2. FUNDAMENTAL PROPERTIES OF HILBERT–KUNZ MULTIPLICITY

Throughout this section, let A be a local ring of characteristic $p > 0$ and let I be an $\mathfrak{m}$-primary ideal of A. In this section, we give several fundamental properties of Hilbert–Kunz multiplicity; see also, e.g., [Mo, HM, Hu, BC, BCP].

If $0 \to L \to M \to N \to 0$ is a short exact sequence of finite A-modules, then

$$e_{\mathrm{HK}}(I, M) = e_{\mathrm{HK}}(I, L) + e_{\mathrm{HK}}(I, N), \tag{2.1}$$

$$e_{\mathrm{HK}}(I, M) \geq 0 \qquad \text{and equality holds if and only if } \dim M < \dim A. \tag{2.2}$$

In particular, we can easily get the formula

$$e_{\mathrm{HK}}(I, M) = \sum_{p \in \mathrm{Assh}_A(A)} e_{\mathrm{HK}}(I, A/p) \cdot l_{A_p}(M_p). \tag{2.3}$$

There are inequalities between the Hilbert–Kunz multiplicity and the usual multiplicity as follows:

$$\max\left\{1, \frac{e(I, A)}{d!}\right\} \leq e_{\mathrm{HK}}(I, A) \leq e(I, A). \tag{2.4}$$

The following formula follows from the so-called Lech's lemma.

$$\text{If } I \text{ is a parameter ideal, then } e_{\mathrm{HK}}(I, A) = e(I, A). \tag{2.5}$$

By definition, one can easily get the following formula. The formula is very simple but it plays an important role in the proof of the Cohen–Macaulay case of Theorem 1.5:

$$e_{\mathrm{HK}}(I^{[q]}, M) = q^d \cdot e_{\mathrm{HK}}(I, M) \quad \text{for all } q = p^e,\ e \geq 1. \tag{2.6}$$

Proof.

$$\begin{aligned} e_{\mathrm{HK}}(I^{[q]}, M) &= \lim_{q' \to \infty} \frac{l_A(M/(I^{[q]})^{[q']}M)}{q'^d} \\ &= q^d \cdot \lim_{q' \to \infty} \frac{l_A(M/(I^{[qq']})M)}{(qq')^d} = q^d \cdot e_{\mathrm{HK}}(I, M). \end{aligned}$$

∎

Hilbert–Kunz multiplicity is very difficult to compute in general and we do not have many examples. But the following formula is very useful and simple for the computation of HK multiplicity of "quotient singularities." In the following, we put $\dim A = d$.

THEOREM 2.7. *Let $(A, \mathfrak{m}) \subset (B, \mathfrak{n})$ be an extension of local domains where B is a finite A-module of rank r and $A/\mathfrak{m} = B/\mathfrak{n}$. Then for every $\mathfrak{m}$-primary ideal I, $e_{\mathrm{HK}}(I, A) = \frac{1}{r} e_{\mathrm{HK}}(IB, B)$.*

In particular, if B is regular, then $e_{\mathrm{HK}}(I) = \frac{1}{r} l_B(B/IB)$.

Proof. Note that

$$e_{\mathrm{HK}}(IB) = \lim_{q\to\infty} \frac{l_B(B/(IB)^{[q]})}{q^d} = \lim_{q\to\infty} \frac{l_A(B/I^{[q]}B)}{q^d} = r \cdot e_{\mathrm{HK}}(I).$$

If B is regular, then $e_{\mathrm{HK}}(I) = l_B(B/IB)$ by (1.4). ∎

Remark. We were informed by Monsky that (2.7) was found independently by Buchweitz, *et al.* [BCP]. We are thankful for them to send the TeX file of their paper to us.

EXAMPLE 2.8. Let A be the rth Veronese subring of $B = k[[X_1, \dots, X_d]]$ ($A = k[[\, m \,|\, m$ is a monomial of degree $r]]$). Then

$$l_B(B/\mathfrak{m}B) = \binom{d+r-1}{r-1} \qquad \text{and} \qquad e_{\mathrm{HK}}(A) = \frac{1}{r}\binom{d+r-1}{r-1}.$$

Since $e(A) = r^{d-1}$, if we fix d and let r tend to ∞, the limit $e(A)/e_{\mathrm{HK}}(A)$ tends to $d!$. Thus the inequality in (2.4) is best possible.

QUESTION 2.9. *Do we always have the strict inequality $e(A)/e_{\mathrm{HK}}(A) < d!$ if $d > 1$?*

The notion of tight closure defined by Hochster and Huneke [HH] is very important in our discussion.

DEFINITION 2.10. (cf. Hochster and Huneke [HH]). Let I be an ideal of A. An element $x \in A$ is said to be in the *tight closure* of I if there exists an element $c \in A^0$ such that for all large $q = p^e$, $cx^q \in I^{[q]}$, where $A^0 := A \setminus \bigcup\{P | P \in \mathrm{Min}(A)\}$. We denote the tight closure of I by I^*.

An ideal I is called *tightly closed* if $I^* = I$. A local ring A in which every ideal (resp. parameter ideal) is tightly closed is called *weakly F-regular* (resp. *F-rational*). Suppose that A is a homomorphic image of a Cohen–Macaulay local ring. Then any F-rational local ring is normal and Cohen–Macaulay. Moreover, A is F-rational if and only if $\mathfrak{q}^* = \mathfrak{q}$ for some parameter ideal $\mathfrak{q}$ of A. See Fedder and Watanabe [FW], Hochster and Huneke [HH], and Huneke [Hu] for more details.

2.11. Let $\bar{I}$ denote the integral closure of I. Then we have $I \subseteq I^* \subseteq \bar{I}$. In general, the integral closure can be characterized in terms of the usual multiplicity. Namely, $\bar{I} = \bar{J}$ implies $e(I, A) = e(J, A)$. If A is quasi-unmixed, then the converse is also true; see, e.g., Rees [Re].

Similarly, Hochster and Huneke [HH, Theorem (8.17)] gave a numerical characterization of tight closure in terms of Hilbert–Kunz multiplicity as follows:

2.12. Let I, J be $\mathfrak{m}$-primary ideals of A with $I \subseteq J$. Then $I^* = J^*$ implies that $e_{\mathrm{HK}}(I, A) = e_{\mathrm{HK}}(J, A)$. The converse is also true provided that A is quasi-unmixed and analytically unramified.

Now let $x \in I$ be an A-regular element and put $\overline{A} = A/xA$ and $\overline{I} = I/xA$. Then it is well known that $e(I, A) \le e(\overline{I}, \overline{A})$. The same holds for Hilbert–Kunz multiplicity.

PROPOSITION 2.13. [Ku2, (3.2)]. *If $x \in I$ is A-regular, then* $e_{\mathrm{HK}}(I, A) \le e_{\mathrm{HK}}(\overline{I}, \overline{A})$.

Proof. Since $x \in I$ is A-regular, we have the exact sequence

$$0 \to A/xA \to A/x^i A \to A/x^{i-1} A \to 0.$$

Applying the functor $- \otimes_A A/I^{[q]}$ to the above sequence, we get

$$A/xA + I^{[q]} \to A/x^i A + I^{[q]} \to A/x^{i-1} A + I^{[q]} \to 0.$$

Hence $l_A(A/I^{[q]}) = l_A(A/x^q A + I^{[q]}) \le q \cdot l_A(A/xA + I^{[q]})$.

Namely, we have

$$\frac{l_A\bigl(A/I^{[q]}\bigr)}{q^d} \le \frac{l_{\overline{A}}\bigl(\overline{A}/(\overline{I})^{[q]}\bigr)}{q^{d-1}} \qquad \text{for all } q = p^e.$$

Let q tend to ∞, and we obtain the required inequality. ∎

In general, if $x(= x_1), x_2, \ldots, x_d$ is a system of parameters which generates a minimal reduction of $\mathfrak{m}$, then $e(x_1, \ldots, x_d, A) = e(A)$. If, in addition, x is A-regular, then we have $e(\overline{A}) = e(x_2, \ldots, x_d, \overline{A}) = e(A)$. However, $e_{\mathrm{HK}}(\overline{A}) = e_{\mathrm{HK}}(A)$ does not necessarily hold. For example, if $A = k[[x, y, z]]/(xy - z^2)$, then $e_{\mathrm{HK}}(A) = 3/2 < 2 \le e_{\mathrm{HK}}(A/aA) = e(A/aA)$ for any A-regular element $a \in \mathfrak{m}$.

2.14. Let $R = \oplus_{n \ge 0} R_n$ be an $\mathbb{N}$-graded ring such that $(R_0, \mathfrak{n})$ is a local ring. Then $N = \mathfrak{n}R + R_+$ is the unique homogeneous maximal ideal. Then we define the Hilbert–Kunz multiplicity of R with respect to a homogeneous ideal J as follows: $e_{\mathrm{HK}}(J, R) := e_{\mathrm{HK}}(JR_N, R_N)$. In particular, we write as $e_{\mathrm{HK}}(R) := e_{\mathrm{HK}}(N, R)$.

Let I be an $\mathfrak{m}$-primary ideal and let t be an indeterminate over A. We have the following graded A-algebras:

$$R(I) := A[It] \subseteq A[t], \qquad \text{where } \deg t = 1,$$

$$G(I) := R(I)/IR(I) \cong \oplus_{n \ge 0} I^n/I^{n+1}.$$

We call $R(I)$ (resp. $G(I)$) *the Rees algebra* (resp. *the associated graded ring*) of I over A. In general, we get $e(I, A) = e(G(I))$ by definition. But in general, $e_{\mathrm{HK}}(I, A) = e_{\mathrm{HK}}(G(I))$ is not true. In general, this equality does not hold. For example, if $A = k[[x, y, z]]/(x^2 + y^3 + z^5)$, where k is a field of characteristic $p \geq 7$, then A is F-rational and thus $e_{\mathrm{HK}}(A) < 2$. On the other hand, since $G := G(\mathfrak{m})$ is isomorphic to a ring $k[X, Y, Z]/(X^2)$, we have $e_{\mathrm{HK}}(G) = e(G) = 2$.

However, we can show that $e_{\mathrm{HK}}(I, A) \leq e_{\mathrm{HK}}(G(I))$ is always true.

THEOREM 2.15. *For any $\mathfrak{m}$-primary ideal I, $e_{\mathrm{HK}}(I, A) \leq e_{\mathrm{HK}}(G(I))$. In particular, we have $e_{\mathrm{HK}}(A) \leq e_{\mathrm{HK}}(G(\mathfrak{m}))$.*

Proof. For simplicity, we put $R = R(I)$ and $G = G(I)$. Since $G_+ = (It, I)R/IR$, we have $G_+^{[q]} = (I^{[q]}t^q, I)R/IR$. Hence $G/G_+^{[q]} = R/(I^{[q]}t^q, I)R$; thus

$$\left[G/G_+^{[q]}\right]_n = \frac{I^n}{I^{[q]}I^{n-q} + I^{n+1}}, \qquad \text{where } I^k = A \text{ for all } k \leq 0.$$

Hence

$$\dim_k \left[G/G_+^{[q]}\right]_n \geq \dim_k \frac{I^n}{I^{[q]} \cap I^n + I^{n+1}} = \dim_k \frac{I^{[q]} + I^n}{I^{[q]} + I^{n+1}}.$$

Fix $q = p^e$. Then $I^{[q]} \supseteq I^n$ for sufficiently large n. Thus $l_G(G/G_+^{[q]}) \geq l_A(A/I^{[q]})$. Taking the limit as $q \to \infty$, we get $e_{\mathrm{HK}}(G_+, G) \geq e_{\mathrm{HK}}(I, A)$. Further, because $G_+^* = \mathfrak{m}G + G_+$, we conclude that $e_{\mathrm{HK}}(G) = e_{\mathrm{HK}}(G_+, G) \geq e_{\mathrm{HK}}(I, A)$ as required. ∎

QUESTION 2.16. *When does the equality $e_{\mathrm{HK}}(I, A) = e_{\mathrm{HK}}(G(I))$ hold ?*

To conclude this section, we propose the following conjecture, which holds if A is a complete intersection local ring; see also Proposition 2.13, Han and Monsky [HM, (5.8)(5.9)] and Dutta [Du].

Conjecture 2.17. Let A be a Cohen–Macaulay local ring of characteristic $p > 0$. Then for any $\mathfrak{m}$-primary ideal I, we have

(1) $e_{\mathrm{HK}}(I, A) \geq l_A(A/I)$.
(2) If $\mathrm{pd}_A\, A/I < \infty$, then $e_{\mathrm{HK}}(I, A) = l_A(A/I)$.

Note that the converse of (2) is not true in general.

3. COLENGTH AND MULTIPLICITY OF TIGHT CLOSURE

We want to show that any unmixed local ring with $e_{\mathrm{HK}}(A) = 1$ is regular. First, we will show that such local rings are Cohen–Macaulay.

It is well-known that for any parameter ideal $\mathfrak{q}$ of a local ring A we have the inequality $e(\mathfrak{q}, A) \leq l_A(A/\mathfrak{q})$ and that equality holds if and only if A is

Cohen–Macaulay. Since $\mathfrak{q} \subseteq \mathfrak{q}^*$, we cannot immediately see which integer is larger $e(\mathfrak{q}, A)$ or $l_A(A/\mathfrak{q}^*)$.

However, for example, let A be an excellent local domain whose normalization B is regular. Then for any parameter ideal $\mathfrak{q}$ of A, we get $\mathfrak{q}^* = \mathfrak{q}B \cap A$; thus we have

$$\begin{aligned} l_A(A/\mathfrak{q}^*) &= l_A(A/\mathfrak{q}B \cap A) = l_A(\mathfrak{q}B + A/\mathfrak{q}B) \\ &\leq l_A(B/\mathfrak{q}B) = e(\mathfrak{q}, B) = e(\mathfrak{q}, A); \end{aligned}$$

see also (4.5). From these observations, we propose the following conjecture.

Conjecture 3.1. Suppose that A is unmixed. Then following statements hold.

(1) For every parameter ideal $\mathfrak{q}$ of A, we have $e(\mathfrak{q}, A) \geq l_A(A/\mathfrak{q}^*)$.

(2) If $e(\mathfrak{q}, A) = l_A(A/\mathfrak{q}^*)$ for some parameter ideal $\mathfrak{q}$ of A, then A is Cohen–Macaulay.

Note that if this is true then one can prove that any unmixed local ring with $e_{\mathrm{HK}}(A) = 1$ is always Cohen–Macaulay. Actually, in Section 4, we show that $e_{\mathrm{HK}}(A) = 1$ implies $e_{\mathrm{HK}}(\mathfrak{q}, A) \leq l_A(A/\mathfrak{q}^*)$.

Before stating our answer to the above conjecture, we need the following definition.

DEFINITION 3.2. Suppose $d = \dim A \geq 1$. Let $\underline{a} = a_1, \dots, a_d$ be a system of parameters of A.

When $d = 1$, we say that $\underline{a} = a_1$ satisfies the condition (#) if $0 : a_1 = 0 : a_1^2$.

When $d \geq 2$, we say that $\underline{a} = a_1, \dots, a_d$ satisfies the condition (#) if the following conditions hold:

(#i) a_i is $A_{i-1}/H^0_{\mathfrak{m}}(A_{i-1})$-regular, where $A_{i-1} = A/(a_1, \dots, a_{i-1})A$ for each $i = 1, \dots, d-1$.

(#ii) $0 :_{A_{d-1}} a_d = 0 :_{A_{d-1}} a_d^2$.

Remark 3.3. (1) There exists at least one system of parameters $\underline{a}$ of A which satisfies (#).

(2) Let $a_1 \in \mathfrak{m}$ be an $A/H^0_{\mathfrak{m}}(A)$-regular element. If $\underline{a} = a_1, \dots, a_d$ satisfies (#), then $\underline{a}' = \overline{a_2}, \dots, \overline{a_d} \in \overline{\mathfrak{m}} := \mathfrak{m}/a_1A$ also satisfies (#). Conversely, if $\overline{a_2}, \dots, \overline{a_d}$ satisfies (#), then we can select $a_2, \dots, a_d \in \mathfrak{m}$ such that $a_1, a_2, \dots, a_d$ satisfies (#).

(3) If A is a Buchsbaum local ring, then every system of parameters satisfies (#).

The following theorem is the main result in this section.

THEOREM 3.4 (See also Proposition 3.9 and Lemma 3.12.). *Let A be an unmixed local ring which is a homomorphic image of a Gorenstein local ring. Then the following statements hold.*

(1) *For every system of parameters $\underline{a}$ of A with (#), $e(\underline{a}, A) \geq l_A(A/(\underline{a})^*)$.*

(2) *Further, if $e(\underline{a}, A) = l_A(A/(\underline{a})^*)$ holds for every system of parameters $\underline{a}$ with (#), then A is Cohen–Macaulay and F-rational.*

The rest of this section is devoted to the proof of the above theorem. From now on, we assume that A is a quasi-unmixed local ring which is a homomorphic image of a Gorenstein local ring of characteristic $p > 0$. Further, let $\underline{a} = a_1, \ldots, a_d$ be a system of parameters of A.

We consider the ideal $\Sigma(\underline{a})$ introduced in [GY],

$$\Sigma(\underline{a}) := \sum_{i=1}^{d}(a_1, \ldots, \widehat{a_i}, \ldots, a_d) : a_i + (a_1, \ldots, a_d)A.$$

To mention the relationship between this ideal and the tight closure, we recall the following theorem, which is called the "*colon capturing property for tight closures*."

THEOREM 3.6 (Hochster and Huneke [HH]). *For any system of parameters $\underline{a}$ of A, we have $(a_1, \ldots, a_{i-1}) : a_i \subseteq (a_1, \ldots, a_{i-1})^*$ for all $i = 1, 2, \ldots, d$.*

By virtue of this theorem, we obtain the following.

COROLLARY 3.7. *Under the same notation as in Theorem 3.6, we have $(\underline{a}) \subseteq \Sigma(\underline{a}) \subseteq (\underline{a})^*$.*

In general, the inequality $e(\underline{a}, A) \geq l_A(A/\Sigma(\underline{a}))$ does not necessarily hold. Actually, when $\dim A = 1$, we have $e(a, A) \geq l_A(A/\Sigma(a)) := l_A(A/aA + 0 : a)$ if and only if $0 : a = 0 : a^2$. However, for every system of parameters $\underline{a}$ with (#), we show that an inequality $e(\underline{a}, A) \geq l_A(A/\Sigma(\underline{a}))$ always holds.

LEMMA 3.8. *If $\underline{a} = a_1, \ldots, a_d$ satisfies (#), then $e(\underline{a}, A) \geq l_A(A/\Sigma(\underline{a}))$.*

Proof. When $d = 1$, since $a_1 \in \mathfrak{m}$ is $A/0 : a_1$-regular, we get

$$e(a_1, A) = e(a_1, A/0 : a_1) = l_A(A/a_1A + 0 : a_1) = l_A(A/\Sigma(\underline{a})).$$

When $d \geq 2$, we put $\overline{A} = A/a_1A$, $\underline{a}' = \overline{a_2}, \ldots, \overline{a_d} \in \overline{\mathfrak{m}} := \mathfrak{m}/a_1A$. Then $\underline{a}'$ also satisfies (#) and

$$\Sigma(\underline{a}', \overline{A}) = \left\{\sum_{i=2}^{d}(a_1, \ldots, \widehat{a_i}, \ldots, a_d) : a_i + (a_1, \ldots, a_d)A\right\} \Big/ a_1A.$$

It follows that $l_A(A/\Sigma(\underline{a})) \leq l_A(\overline{A}/\Sigma(\underline{a}', \overline{A})) \leq l_A(A_{d-1}/(0 : a_d + a_d A_{d-1}))$.

On the other hand, by the choice of a_1, we have $e(\underline{a}', 0 : a_1) = 0$; in fact, $l_A(0 : a_1) < \infty$ and $\dim \overline{A} = d - 1 \geq 1$. Hence $e(\underline{a}, A) = e(\underline{a}', \overline{A}) = \cdots = e(a_d, A_{d-1})$. By the similar reasoning as in case of $d = 1$, we obtain an equality $e(a_d, A_{d-1}) = l_A(A_{d-1}/(0 : a_d + a_d A_{d-1}))$. Thus we get the required inequality

$$e(\underline{a}, A) - l_A(A/\Sigma(\underline{a})) \geq e(\underline{a}', \overline{A}) - l_A(\overline{A}/\Sigma(\underline{a}', \overline{A})) \geq 0.$$

∎

As an application of this lemma, we get

PROPOSITION 3.9. *For every system of parameters* $\underline{a}$ *of* A *with* (#), *we have*

$$e(\underline{a}, A) \geq l_A(A/(\underline{a})^*).$$

We say that A satisfies the Serre condition (S_n) if $\operatorname{depth} A_P \geq \min\{n, \dim A_P\}$ for every prime ideal P of A. A system of parameters $\underline{a} = a_1, \ldots, a_d$ of A is said to be *standard* if the following equation holds:

$$l_A(A/(\underline{a})) - e(\underline{a}, A) = \sum_{i=0}^{d-1} \binom{d-1}{i} l_A(H^i_{\mathfrak{m}}(A)).$$

In particular, if A admits a standard s.o.p. then it is (F.L.C.), that is, $l_A(H^i_{\mathfrak{m}}(A)) < \infty$ for all $i \neq d$. Moreover, it is well known that standard the s.o.p of A is generated by the so-called *unconditioned strong d-sequence*; see, e.g., Schenzel [Sch2]. Hence we remark that any standard s.o.p satisfies (#).

EXAMPLE 3.10. (cf. Goto [Go, Sects. 3,4] and Goto–Yamagishi [GY, Proposition (3.15)]. Let A be (F.L.C.) with $d := \dim A \geq 2$, and let $\underline{a}$ is a standard s.o.p. of A. Then $\underline{a}$ satisfies (#) and we get

$$e(\underline{a}, A) = l_A(A/\Sigma(\underline{a})) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(H^i_{\mathfrak{m}}(A)).$$

In particular, $e(\underline{a}, A) = l_A(A/\Sigma(\underline{a}))$ if and only if $A/H^0_{\mathfrak{m}}(A)$ is Cohen–Macaulay.

The following is a key result in this section.

THEOREM 3.11. *Let A be a local ring with $d := \dim A \geq 2$. Suppose that the following two conditions hold*:

(1) *$A/H^0_{\mathfrak{m}}(A)$ is unmixed.*

(2) *$e(\underline{a}, A) = l_A(A/\Sigma(\underline{a}))$ for every system of parameters $\underline{a}$ which satisfies* (#).

Then $A/H^0_{\mathfrak{m}}(A)$ is Cohen–Macaulay.

In order to prove the above theorem, we need the following lemma.

LEMMA 3.12. (cf. Flenner [F, Lemma (3.2)]). *Let n be a given non-negative integer. Assume that the following two conditions hold.*

(1) *A_P satisfies (S_n) for any prime $P \in \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$.*

(2) *(S_{n+1})-locus $\mathcal{U}_{n+1}$ is open in $\operatorname{Spec}(A)$.*

Then $\mathcal{F}_n := \{P \in \operatorname{Spec}(A) \setminus \{\mathfrak{m}\} \mid \operatorname{depth} A_P = n < \dim A_P\}$ is a finite set.

Remark (cf. Grothendieck [Gr, Chap. IV, No. 24]). Let A be an excellent local ring. Then for any positive integer n, (S_n)-locus $\mathcal{U}_n := \{P \in \operatorname{Spec}(A) | A_P$ satisfies $(S_n)\}$ of A is open in $\operatorname{Spec}(A)$.

Proof of Proposition 3.11. Let A be a local ring which satisfies the assumption. Our proof works by induction on $d := \dim A \geq 2$. The condition (1) implies that A is quasi-unmixed. We put $H = H^0_{\mathfrak{m}}(A)$, $B = A/H$.

First suppose $d = 2$. Since A is (F.L.C.) by assumption, we can take a standard system of parameters $\underline{a} = a_1, a_2$ of A. Then we note that $\underline{a}$ satisfies (#). Thus we get

$$e(\underline{a}, A) = l_A(A/\Sigma(\underline{a})) \tag{3.11.1}$$

by our assumption and

$$e(\underline{a}, A) = l_A(A/\Sigma(\underline{a})) + l_A(H^1_{\mathfrak{m}}(A)) \tag{3.11.2}$$

by Example 3.10. Comparing (3.11.1) with (3.11.2), we get $H^1_{\mathfrak{m}}(A) = 0$; thus $B = A/H$ is Cohen–Macaulay.

Next suppose $d \geq 3$. As A is quasi-unmixed, by Aoyama and Goto [AG], the (S_2)-locus $\mathcal{U}_2$ of A is open in $\operatorname{Spec}(A)$. Applying Flenner's lemma to the case $n = 1$, we see that $\mathcal{F}_1 = \{P \in \operatorname{Spec}(A) \setminus \{\mathfrak{m}\} \mid \operatorname{depth} A_P = 1 < \dim A_P\}$ is a finite set. Hence we can take an element $a_1 \in \mathfrak{m}$ which does not belong to any prime $P \in \operatorname{Min}(A) \cup \mathcal{F}_1$. In particular, a_1 is $A/H^0_{\mathfrak{m}}(A)$-regular.

Put $\overline{A} = A/a_1A$. Then $\overline{A}$ is also a quasi-unmixed local ring.

CLAIM 1. *For any system of parameters $\underline{a}' = \overline{a_2}, \ldots, \overline{a_d} \in \overline{\mathfrak{m}}$ which satisfies* (#), *we have $e(\underline{a}', \overline{A}) = l_A(\overline{A}/\Sigma(\underline{a}'))$.*

Actually, since $\underline{a} = a_1, a_2, \ldots, a_d$ satisfies (#), we have $e(\underline{a}, A) = l_A(A/\Sigma(\underline{a}))$ by our assumption. Then as in the proof of Lemma (3.8), we obtain equality $e(\underline{a}', \overline{A}) = l_A(\overline{A}/\Sigma(\underline{a}'))$ as required.

CLAIM 2. *$\overline{A}/H^0_{\mathfrak{m}}(\overline{A})$ is unmixed.*

It is enough to show that $\mathrm{Ass}(\overline{A}) \subseteq \mathrm{Min}(A) \cup \{\overline{\mathfrak{m}}\}$. Actually, for any prime $\overline{P} := P/a_1A \in \mathrm{Spec}(\overline{A}) \setminus \{\overline{\mathfrak{m}}\}$, since a_1 is A_P-regular, we have

$$\operatorname{depth} \overline{A}_{\overline{P}} = \operatorname{depth} A_P - 1 \quad \text{and} \quad \dim \overline{A}_{\overline{P}} = \dim A_P - 1.$$

Suppose $\overline{P} \in \mathrm{Ass}(\overline{A})$. Then $\operatorname{depth} A_P = 1$. By the choice of a_1, we get $P \notin \mathcal{F}_1$; thus $\dim A_P \le 1$. It follows that $\overline{P} \in \mathrm{Min}(\overline{A})$.

According to Claim 1 and Claim 2, $\overline{A}$ also satisfies the assumption of the proposition. Thus by the induction hypothesis, $\overline{A}/H^0_{\mathfrak{m}}(\overline{A})$ is Cohen–Macaulay; that is, $H^i_{\mathfrak{m}}(\overline{A}) = 0$ for all $i = 1, \ldots, d-2$.

On the other hand, since $0 : a_1$ is a module of finite length, we get the following long exact sequence of local cohomology modules:

$$\begin{aligned} 0 \to 0\colon a_1 \to H^0_{\mathfrak{m}}(A) &\xrightarrow{a_1} H^0_{\mathfrak{m}}(A) \to H^0_{\mathfrak{m}}(\overline{A}) \\ &\to H^1_{\mathfrak{m}}(A) \xrightarrow{a_1} H^1_{\mathfrak{m}}(A) \to H^1_{\mathfrak{m}}(\overline{A}) \\ &\to \cdots \\ &\to H^{d-1}_{\mathfrak{m}}(A) \xrightarrow{a_1} H^{d-1}_{\mathfrak{m}}(A) \to H^{d-1}_{\mathfrak{m}}(\overline{A}). \end{aligned}$$

Since $H^{i-1}_{\mathfrak{m}}(\overline{A}) = 0$ for all $i = 2, \ldots, d-1$, $0 \to H^i_{\mathfrak{m}}(A) \xrightarrow{a_1} H^i_{\mathfrak{m}}(A)$ is exact. This implies that $H^i_{\mathfrak{m}}(A) = 0$ for all $i = 2, \ldots, d-1$, because $H^i_{\mathfrak{m}}(A)$ is Artinian.

In order to complete the proof of the proposition, we must show that $H^1_{\mathfrak{m}}(A) = 0$. Since A/H is unmixed, $M := \mathrm{Hom}_A(H^1_{\mathfrak{m}}(A), E_A)$ is a module of finite length; thus so is $H^1_{\mathfrak{m}}(A)$. In fact, for any prime $P \in \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$, $M_P \neq 0$ if and only if $H^{1-\dim A/P}_{PA_P}(A_P) \neq 0$ by the local duality theorem. The latter condition implies that $P \in \mathrm{Ass}(A)$ with $\dim A/P = 1$ and since such a prime ideal P does not exist by the assumption (1), M is a module of finite length.

Further, $H^1_{\mathfrak{m}}(\overline{A}) = 0$ gives an exact sequence $H^1_{\mathfrak{m}}(A) \xrightarrow{a_1} H^1_{\mathfrak{m}}(A) \to 0$. Hence by Nakayama's lemma, we have $H^1_{\mathfrak{m}}(A) = 0$ as required.

Therefore we conclude that $A/H^0_{\mathfrak{m}}(A)$ is Cohen–Macaulay. ∎

Remark. We are thankful to Yanagawa for advice about the above proof.

By Corollary 3.7 and Theorem 3.11, we obtain the following.

THEOREM 3.13. *Let A be an unmixed local ring of characteristic $p > 0$. If $e(\underline{a}, A) = l_A(A/(\underline{a})^*)$ for every system of parameters $\underline{a}$ of A with (#), then A is Cohen–Macaulay and F-rational.*

Our Proposition 3.11 is false without the assumption that $A/H^0_{\mathfrak{m}}(A)$ is unmixed.

EXAMPLE 3.14. Let $A = k[[X, Y, Z]]/(XY, XZ)$. Then $\dim A = 2$ and A is not equi-dimensional. Put $a_1 = x^n + y$ and $a_2 = x^n + z$. Then $\underline{a} = a_1, a_2$ is a system of parameters which satisfies (#). Moreover, the following statements hold:

(1) $e(\underline{a}, A) = l_A(A/\Sigma(\underline{a})) = 1$. In fact, $\Sigma(\underline{a}) = (x, y, z)A$.

(2) $(\underline{a})^* = (x^n, y, z)$. In particular, $l_A(A/(\underline{a})^*) = n$.

Thus "colon capturing" does not hold in this example.

EXAMPLE 3.15. Let $A = k[[X, Y, Z]]/(X^2, XY)$. Then A is quasi-unmixed with depth $A = 1 > 0$ but not unmixed. Put $a_1 = z$ and $a_2 = y$. Then $\underline{a} = a_1, a_2$ satisfies (#). Moreover, the following statements hold.

(1) $e(\underline{a}, A) = l_A(A/\Sigma(\underline{a})) = l_A(A/(\underline{a})^*) = 1$.

(2) y, z does not satisfy (#). Moreover, $e(y, z; A) = 1 < 2 = e(z; A/yA)$.

4. PROOF OF THEOREM 1.5

Throughout this section, let A be a local ring of characteristic $p > 0$ and let I be an $\mathfrak{m}$-primary ideal of A. In the previous section, we had several inequalities between colength and multiplicity of tight closure. Using the inequalities, we can show that an unmixed local ring A with $e_{\mathrm{HK}}(A) = 1$ is regular (see Theorem 1.5). We begin with the following result, which is a starting point of our investigation.

PROPOSITION 4.1. *Suppose that* $e_{\mathrm{HK}}(A) = 1$. *Then* $e(\mathfrak{q}, A) \le l_A(A/\mathfrak{q}^*)$ *for every parameter ideal* $\mathfrak{q}$ *of* A.

The proposition follows easily from the following lemma; see also (2.5).

LEMMA 4.2. *The following statements hold*:

(1) $e_{\mathrm{HK}}(I, A) \le l_A(A/I^*) \cdot e_{\mathrm{HK}}(A)$.

(2) *If* $I \subseteq J$, *then* $e_{\mathrm{HK}}(I, A) \le e_{\mathrm{HK}}(J, A) + l_A(J/I) \cdot e_{\mathrm{HK}}(A)$.

Proof. Take a composition series of A/I^* as

$$A = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_r = I^*, \qquad \text{where } I_{i-1}/I_i \cong A/\mathfrak{m} \text{ and } r = l_A(A/I^*).$$

Then we get

$$A = I_0^{[q]} \supseteq I_1^{[q]} \supseteq \cdots \supseteq I_r^{[q]} = (I^*)^{[q]} \qquad \text{for all } q = p^e,\ e \ge 1.$$

Since $I_{i-1}^{[q]}/I_i^{[q]}$ is a homomorphic image of $A/\mathfrak{m}^{[q]}$, we have

$$l_A\Big(A/(I^*)^{[q]}\Big) = \sum_{i=1}^{r} l_A\Big(I_{i-1}^{[q]}/I_i^{[q]}\Big) \le \sum_{i=1}^{r} l_A\Big(A/\mathfrak{m}^{[q]}\Big)$$
$$= l_A(A/I^*)\cdot l_A\Big(A/\mathfrak{m}^{[q]}\Big).$$

It follows that $e_{\mathrm{HK}}(I, A) = e_{\mathrm{HK}}(I^*, A) \le l_A(A/I^*)e_{\mathrm{HK}}(A)$.

In order to prove the second statement, we may assume that $l_A(J/I) = 1$. Then we can obtain the following inequality:

$$l_A(A/I^{[q]}) = l_A(A/J^{[q]}) + l_A(J^{[q]}/I^{[q]}) \le l_A(A/J^{[q]}) + l_A(A/\mathfrak{m}^{[q]}).$$

Hence we get $e_{\mathrm{HK}}(I, A) \le e_{\mathrm{HK}}(J, A) + e_{\mathrm{HK}}(A)$. ∎

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let A be an unmixed local ring with $e_{\mathrm{HK}}(A) = 1$. Since $e_{\mathrm{HK}}(\widehat{A}) = e_{\mathrm{HK}}(A) = 1$, we may assume that A is complete.

First, we show that A is Cohen–Macaulay. Take any system of parameters $\underline{a} = a_1, \ldots, a_d$ of A which satisfies (#). Then by Proposition (3.9), we have $e(\underline{a}, A) \ge l_A(A/(\underline{a})^*)$. Further, by Proposition 4.1, we have $e(\underline{a}, A) = l_A(A/(\underline{a})^*)$. Thus it follows from Theorem 3.13 that A is Cohen–Macaulay.

Next, we show that $e_{\mathrm{HK}}(I, A) = l_A(A/I)$ for any $\mathfrak{m}$-primary ideal I of A. Take a parameter ideal $\mathfrak{q}$ with $\mathfrak{q} \subseteq I$. Since A is Cohen–Macaulay, we have $l_A(A/\mathfrak{q}) = e(\mathfrak{q}, A) = e_{\mathrm{HK}}(\mathfrak{q}, A)$.

On the other hand, according to the previous proposition, we can get

$$e_{\mathrm{HK}}(I, A) \le l_A(A/I^*) \le l_A(A/I)$$

and

$$e_{\mathrm{HK}}(\mathfrak{q}, A) \le e_{\mathrm{HK}}(I, A) + \Big\{ l_A(A/\mathfrak{q}) - l_A(A/I) \Big\}.$$

It follows that $e_{\mathrm{HK}}(I, A) = l_A(A/I)\big(= l_A(A/I^*)\big)$.

In particular, substituting $\mathfrak{m}^{[q]}$ for I, we get

$$l_A(A/\mathfrak{m}^{[q]}) = e_{\mathrm{HK}}(\mathfrak{m}^{[q]}, A) = q^d \cdot e_{\mathrm{HK}}(\mathfrak{m}, A) = q^d$$
$$\textit{for all } q = p^e,\ e \ge 1.$$

By Kunz' theorem, we conclude that A is regular. ∎

In general, the condition $e_{\mathrm{HK}}(A) = 1$ does not imply that A is unmixed.

EXAMPLE 4.3. Let $A = k[[X_1, \ldots, X_d, Y]]/(YX_1, \ldots, YX_d, Y^2)$, where k is a field of characteristic $p > 0$ and $d = \dim A \ge 1$. Put

$H := yA = H^0_{\mathfrak{m}}(A)$. Then for any $\mathfrak{m}$-primary ideal I of A, we have $I^* = I + H$ and $l_A(A/I^*) = e_{\mathrm{HK}}(I, A)$. In particular, $e_{\mathrm{HK}}(A) = 1$.

EXAMPLE 4.4. Nagata [Na, Appendix, Example 2] gave an example of a local domain A which is not unmixed such that

$$\widehat{A} \cong k[[X, Y, Z]]/(XY, XZ).$$

Then we have $e(A) = e_{\mathrm{HK}}(A) = 1$.

There are many examples in which Conjecture 3.1 is true.

EXAMPLE 4.5. Let A be an equi-dimensional local ring of characteristic $p > 0$. Assume that there exists a Noetherian ring B such that

(i) $A \subseteq B$ and

(ii) B is a finite Cohen–Macaulay A-module.

Then for any parameter ideal $\mathfrak{q}$ of A, we have $e(\mathfrak{q}, A) \geq l_A(A/\mathfrak{q}^*)$. Furthermore, if equality holds for some $\mathfrak{q}$, then A is Cohen–Macaulay and F-rational.

In particular, the conditions are satisfied if A is an excellent equi-dimensional local ring with $\dim A \leq 2$.

Proof. By our assumption, $\mathfrak{q}^* = (\mathfrak{q}B)^* \cap A \supseteq \mathfrak{q}B \cap A$ for any parameter ideal $\mathfrak{q}$ of A. Thus we get

$$\begin{aligned} l_A(A/\mathfrak{q}^*) &\leq l_A(A/\mathfrak{q}B \cap A) = l_A(A + \mathfrak{q}B/\mathfrak{q}B) \\ &= l_A(B/\mathfrak{q}B) - l_A(B/\mathfrak{q}B + A). \end{aligned}$$

Since B is a maximal Cohen–Macaulay A-module, we have

$$l_A(B/\mathfrak{q}B) = e(\mathfrak{q}, B) \qquad \text{and} \qquad l_A(B/A + \mathfrak{q}B) \geq e(\mathfrak{q}, B/A).$$

Hence $l_A(B/\mathfrak{q}B) - l_A(B/\mathfrak{q}B + A) \leq e(\mathfrak{q}, B) - e(\mathfrak{q}, B/A) = e(\mathfrak{q}, A)$. In particular, we have $l_A(A/\mathfrak{q}^*) \leq e(\mathfrak{q}, A)$.

Now suppose $l_A(A/\mathfrak{q}^*) = e(\mathfrak{q}, A)$. It follows that B/A is Cohen–Macaulay and hence so is A. Moreover, since $l_A(A/\mathfrak{q}) = e(\mathfrak{q}, A) = l_A(A/\mathfrak{q}^*)$ and $\mathfrak{q} \subseteq \mathfrak{q}^*$, we get $\mathfrak{q} = \mathfrak{q}^*$. Thus A is Cohen–Macaulay and F-rational. ∎

5. LOCAL RINGS OF SMALL HILBERT–KUNZ MULTIPLICITY

5.1. Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d of characteristic $p > 0$. We want to characterize A with small $e_{\mathrm{HK}}(A)$.

We fix a minimal reduction J of $\mathfrak{m}$. Since A is Cohen–Macaulay, the associated graded ring $G(J) \cong (A/J)[X_1, \ldots, X_d]$ and

$$G(J/J^{[q]}) \cong (A/J)[X_1, \ldots, X_d]/(X_1^q, \ldots, X_d^q),$$

we can compute $l_A(A/J^n)$ by counting the number of monomials in a polynomial ring.

Now, fix the multiplicity $e(A) = l_A(A/J)$ of A and $d = \dim A$. We want to get a good lower bound of $e_{\mathrm{HK}}(A)$ in this class. For that purpose, we want to find a good upper bound for $l_A(\mathfrak{m}^{[q]}/J^{[q]})$ since $e_{\mathrm{HK}}(A) = e(A) - \lim_{q\to\infty} l_A(\mathfrak{m}^{[q]}/J^{[q]})/q^d$.

Since we are interested in Hilbert–Kunz multiplicity, we introduce the following convention.

5.1.1. Let $f(q)$ and $g(q)$ be polynomials of q with degree $\leq d$. Then we write $f(q) \sim g(q)$ if $\deg(f(q) - g(q)) < d$. Also, if $I(q)$ and $I'(q)$ are $\mathfrak{m}$-primary ideals depending on q, we write $I(q) \sim I'(q)$ if $l_A(A/I(q)) \sim l_A(A/I'(q))$. Also, for $s \in \mathbb{Q}$, sq will be some integer near to sq.

If we calculate $l_A(A/I(q))$ which is a polynomial of q of degree d for $q \gg 0$, we agree to neglect lower order terms and discuss only the coefficient of q^d.

EXAMPLE 5.1.2. Since $\mathfrak{m}^n = J^r\mathfrak{m}^{n-r}$ for some r and every $n \geq r$, $\mathfrak{m}^{sq} \sim J^{sq}$ for every positive rational number s. Also, we have $l_A(A/\mathfrak{m}^{sq}) \sim (s^d q^d/d!)e(A)$ if $s \geq 1$.

The following statement is contained in [BC, BCP] with different proof.

LEMMA 5.2. *If A is a hypersurface of degree e and if $2 \leq d \leq 6$, $e_{\mathrm{HK}}(A) \geq c(d)e$, where we put*

$$c(2) = \tfrac{3}{4}, \qquad c(3) = \tfrac{2}{3}, \qquad c(4) = \tfrac{115}{192}, \qquad c(5) = \tfrac{11}{20}, \qquad c(6) = \tfrac{5633}{11{,}520}.$$

(*In* [*BC*, *Theorem* 5], *it is shown that this bound is best possible in dimension* 2 *and* 3, *but not in dimension* 4 *by the computation of Monsky*.)

Proof. Let $\mathfrak{m} = (u, J)$. Then, since $u^q \in \mathfrak{m}^q$,

$$l_A(\mathfrak{m}^{[q]}/J^{[q]}) \leq l_A(A/\mathfrak{m}^{q(d-1)/2}) + l_A((\mathfrak{m}^{q(d+1)/2} + J^{[q]})/J^{[q]}).$$

Since we are only interested in terms of degree d in q, we may replace $\mathfrak{m}$ by J. Then since the associated graded ring $G(J) \cong A/J[X_1, \ldots, X_d]$, we have $l_A((J^{q(d+1)/2} + J^{[q]})/J^{[q]}) = l_A(A/J^{q(d-1)/2})$ by duality in $A/J^{[q]}$.

Now, $\lim_{q\to\infty} \dim_k(k[X_1, \ldots, X_d]/(X_1, \ldots, X_d)^{q(d-1)/2})/q^d$ is equal to the volume of the subset of the d-dimensional cube given by the equation

$$\left\{(x_1, \ldots, x_d)\middle| 0 \le x_i \le 1 \text{ for every } i \text{ and } \sum_{i=1}^{d} x_i \le \frac{d-1}{2}\right\}$$

and is given by $(1 - c(d))/2$ for $2 \le d \le 6$. Hence we conclude $l_A(A/\mathfrak{m}^{q(d-1)/2}) + l_A((\mathfrak{m}^{q(d+1)/2} + J^{[q]})/J^{[q]}) \sim (1 - c(d))q^d e$ for $2 \le d \le 6$. ∎

In the following, we study Cohen–Macaulay two-dimensional local ring A with $e_{\mathrm{HK}}(A) \le 2$. First, note the following fact [Hu].

5.3. Assume A is Cohen–Macaulay with $\dim A = d$ and $e(A) = 2$. Then A is F-rational if and only if $e_{\mathrm{HK}}(A) < 2$.

Proof. By our assumption, we can take a parameter ideal J with $l_A(A/J) = 2$. Then by (2.12), $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(\mathfrak{m}) < e_{\mathrm{HK}}(J) = e(A) = 2$ if and only if A is F-rational. If A is not F-rational, $\mathfrak{m} = J^*$ and we have $e_{\mathrm{HK}}(\mathfrak{m}) = e_{\mathrm{HK}}(J^*) = e_{\mathrm{HK}}(J) = e(A) = 2$. ∎

It is known that rational double points defined over $\mathbb{C}$ are quotient singularities of finite subgroups of $G \subset SL(2, \mathbb{C})$.

By Theorem (2.7), $|G|$ is essential to compute $e_{\mathrm{HK}}(A)$.

THEOREM 5.4. *Let A be Cohen–Macaulay with* $\dim A = 2$. *Then*

(1) $1 < e_{\mathrm{HK}}(A) < 2$ *if and only if A is an F-rational double point. In this case,* $e_{\mathrm{HK}}(A) = 2 - \frac{1}{|G|}$, *where G is the finite subgroup of $SL(2, k)$ attached to the corresponding singularity in characteristic* 0.

(2) $e_{\mathrm{HK}}(A) = 2$ *if and only if A is either a non-F-rational double point or A is the "ordinary triple point"* ($\widehat{A} \cong k[[x^3, x^2y, xy^2, y^3]]$).

Proof. Since completion and the extension of the base field do not change Hilbert–Kunz multiplicity, we may assume A is complete and that the residue field of A is algebraically closed. If A is an F-rational double point, then A is a rational double point by [Sm] and their equations are classified by Artin [A2].

If A is of (A_n) type, then $A \cong k[[x, y, z]]/(xy - z^{n+1})$ and we can easily show that $e_{\mathrm{HK}}(A) = 2 - \frac{1}{n+1}$ by either Theorem 2.7 or Conca [C].

If A is of type (D_n) (resp. (E_6), (E_7), resp. (E_8)), then A is F-rational implies $p > 2$ (resp. $p > 3$, resp. $p > 5$) and $A \cong \widehat{B}$, where $B = k[x, y]^G$ is the invariant subring of a finite subgroup G of $SL(2, k)$ acting on $k[x, y]$. Then it remains to use Theorem 2.7 noting that we can take $B = k[f, g, h]$ so that f, g are homogeneous polynomials and h is the Jacobian of f and g. It follows that h is a generator of the socle of $k[x, y]/(f, g)$. Since the length of $k[x, y]/(f, g)$ is $2|G|$, we have $e_{\mathrm{HK}}(A) = 2 - \frac{1}{|G|}$. ∎

To proceed further, we need another lemma.

LEMMA 5.5. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring with* $\dim A = 2$ *and let J be a minimal reduction of $\mathfrak{m}$. If $\mathfrak{m}/J$ is generated by r elements, then*

$$e_{\mathrm{HK}}(A) \geq \frac{r+2}{2(r+1)} e(A).$$

Proof. Let $u_1, \ldots, u_r$ be generators of $\mathfrak{m}/J$. Then we have

$$\begin{aligned} l_A(\mathfrak{m}^{[q]}/J^{[q]}) &\leq l_A\left(\frac{\mathfrak{m}^{[q]} + \mathfrak{m}^{(r+2)q/(r+1)}}{J^{[q]} + \mathfrak{m}^{(r+2)q/(r+1)}}\right) + l_A\left(\frac{\mathfrak{m}^{(r+2)q/(r+1)} + J^{[q]}}{J^{[q]}}\right) \\ &\leq \left(\frac{r}{2} \cdot \frac{q^2}{(r+1)^2} + \frac{1}{2} \cdot \frac{r^2 q^2}{(r+1)^2}\right) e(A) = \frac{r}{2r+2} q^2 e(A). \end{aligned}$$

∎

By this lemma, since $\mathfrak{m}/J$ is generated by at most $e(A) - 1$ elements, we have

$$e_{\mathrm{HK}}(A) \geq \frac{e(A)+1}{2}. \tag{5.5.1}$$

Note that we have equality in (5.5.1) when (and only when ?) A is the $(r+1)$th Veronese subring of $k[x, y]$ by (2.8).

Let us continue the proof of Theorem 5.4. Since we are interested in the rings with $e_{\mathrm{HK}}(A) \leq 2$ and $e_{\mathrm{HK}}(A) \geq 5/2$ if $e(A) \geq 4$, it remains to study rings with $e(A) = 3$. By Lemma 5.2, if A is a hypersurface, then $e_{\mathrm{HK}}(A) \geq 9/4 > 2$. If A is not F-rational, then $\mathfrak{m}/J^*$ is generated by at most one element and again we have $e_{\mathrm{HK}}(A) \geq 9/4 > 2$.

Thus we may assume that A is F-rational and hence is a rational singularity (cf. [Sm]). Now, the "graphs" of resolution of rational triple points are classified in [A1]. Let $\phi: X \to \operatorname{Spec}(A)$ be the resolution of $\operatorname{Spec}(A)$ and let Z be the fundamental cycle of X (the minimal positive cycle with $ZE \leq 0$ for every irreducible exceptional curve E). It is shown in [A1] that $H^0(X, \mathcal{O}_X(-nZ)) = \mathfrak{m}^n$ and $\mathfrak{m}^n \mathcal{O}_X = \mathcal{O}_X(-nZ)$ for every positive integer n.

Until the end of this section, let A be a two-dimensional rational triple point.

Now, if Z is not reduced, it is easy to show by classification of the graphs that there is an irreducible exceptional curve E on X with $E^2 = -2$, $EZ = -1$, and $Z \geq 2E$. Then by the exact sequence

$$0 \to \mathcal{O}_X(-2Z) \to \mathcal{O}_X(-2Z+E) \to \mathcal{O}_E \to 0,$$

we conclude that there is $f \in H^0(X, \mathcal{O}_X(-2Z+E))$, $f \notin \mathfrak{m}^2 = H^0(X, \mathcal{O}_X(-2Z))$. Then we have $f^2 \in \mathfrak{m}^3$ since $2(2Z - E) \geq 3Z$. Then by the argument above, $e_{\mathrm{HK}}(A) \geq 9/4$ in this case.

If the fundamental cycle Z is reduced, then the singularity is parametrized by three natural numbers a, b, c with $a \le b \le c$ so that

$$E := \phi^{-1}(\mathfrak{m}) = E_0 \cup \bigcup_{j=1}^{a-1} E_{1,j} \cup \bigcup_{j=1}^{b-1} E_{2,j} \cup \bigcup_{j=1}^{c-1} E_{3,j},$$

where $E_0^2 = -3$, $E_{i,j}^2 = -2$ for every i, j (if $a = 1$, there exists no $E_{1,j}$, etc.), E_0 (resp. $E_{i,1}$, resp. $E_{i,j}$ $(j > 1)$) intersects only $E_{i,1}$ (resp. E_0 and $E_{i,2}$, resp. $E_{i,j-1}$ and $E_{i,j+1}$).

Now, consider the filtration $F^l = H^0(X, \mathcal{O}(-lE_0))$ on A. Then it is shown in [TW1, 6.3] that the associated graded ring

$$G_F(A) := \bigoplus_{l \ge 0} F^l/F^{l+1} \cong k\left[T, xT^a, x^{-1}T^b, \frac{1}{x+1}T^c\right] \subset k(x)[T]. \quad (5.5.2)$$

First, we will compute the HK multiplicity of the rings in (5.5.2).

LEMMA 5.6. *Let $R = k\left[T, xT^a, x^{-1}T^b, \frac{1}{x+1}T^c\right]$ and let A be the localization of R at the unique graded maximal ideal. Then*

$$e_{\mathrm{HK}}(A) = 3 - \frac{a+b+c}{ab+bc+ca}.$$

Hence $e_{\mathrm{HK}}(A) = 2$ if $(a, b, c) = (1, 1, 1)$, $e_{\mathrm{HK}}(A) = \frac{11}{5}$ if $(a, b, c) = (1, 1, 2)$, and $e_{\mathrm{HK}}(A) \ge \dfrac{9}{4}$ otherwise.

Proof. We put $t = T$, $u = xT^a$, $v = x^{-1}T^b$, $w = \frac{1}{(x+1)}T^c$ so that we have $\mathfrak{m} = (t, u, v, w)$ with relations

$$uv = t^{a+b}, uw = t^{a+c} - t^a w, vw = t^c v - t^b w. \quad (5.6.1)$$

We put $R(q) := A/\mathfrak{m}^{[q]} \cong R/(t^q, u^q, v^q, w^q)$. $R(q)$ is a graded ring with $\deg t = 1$, $\deg u = a$, $\deg v = b$, and $\deg w = c$ and generated as a k vector space by three sets of generators $\{t^i u^j | 0 \le i < q, 0 \le j < q\}$, $\{t^i v^j | 0 \le i < q, 0 < j < q\}$, and $\{t^i w^j | 0 \le i < q, 0 < j < q\}$.

We discuss the relations among these generators. We always adopt the convention (5.1.1). So never mind if the number of elements is not an integer. They will come up to the right answer after the limit is taken.

Now, $0 = v^i u^q = t^{(a+b)i} u^{q-i}$. Also since $0 = w^i u^q$ in $R(q)$ and $\frac{x^q}{(x+1)^i} = \frac{((x+1)-1)^q}{(x+1)^i}$, $0 = (u + t^a)^{q-i} - t^{aq} w^i$. Thus we have $\frac{n}{b} + \frac{n}{c}$ relations in degree $aq + n$. These relations are independent until $\frac{n}{b} + \frac{n}{c} \le \frac{q-n}{a}$, the latter being the number of bases of the form $t^i u^j$, $i < q$, $j < q$ and degree $aq + n$. Summing up, we check that there are $[bc(b+c)/2(ab+bc+ca)^2]q^2$ relations in degrees aq to $aq + \frac{bcq}{ab+bc+ca}$. Also, there are $a(b+c)^2 q^2/2(ab+bc+ca)^2$ monomials of the form $t^i u^j$ with degree larger than $aq + \frac{bcq}{ab+bc+ca}$, which are 0 in $R(q)$. Summing up, there are $\frac{b+c}{2(ab+bc+ca)}q^2$ independent relations among the monomials of the first type. Doing the same thing to the other two types, we see that there are $(a+b+c)q^2/ab+bc+ca$ independent relations among $3q^2$ monomials and hence we have the desired result.

Remark 5.7. In the case of analytic local rings, it is shown by Tyurina that rational triple points are rigid. That is, $\widehat{A}$ is isomorphic to the completion of a graded ring with the same graph. If this is also the case in every positive characteristic, the proof of Lemma 5.5 is complete by Lemma 5.6. But since we do not know if the proof is valid for any characteristic or not, we continue the proof.

Now, it is easy to see that $G(\mathfrak{m})$ is an integral domain if and only if $(a, b, c) = (1, 1, 1)$. In this case, we can take minimal generating system (x, y, z, w) of $\mathfrak{m}$ with $y^2 - xz = z^2 - yw = yz - xw = 0$, $\widehat{A} \cong k[[s^3, s^2t, st^2, t^3]] \subset k[[s, t]]$, and $e_{\mathrm{HK}}(A) = 2$.

It remains to show that if $c > 1$, then $e_{\mathrm{HK}}(A) > 2$. If $a = 1$, then E is a chain and A is a "cyclic quotient singularity," that is, A has a cyclic cover which is regular local ring and thus $\widehat{A} \cong \widehat{G_F(A)}$. In this case, $e_{\mathrm{HK}}(A)$ is computed in Lemma 5.6.

If $a > 1$, then by our argument above, $\mathfrak{m}$ is generated by (t, u, v, w) with $uv, uw, vw \in t^2A \cap \mathfrak{m}^3$. We can take $J = (t, s)$ with $s = u + v + w$.

Now, since $u^2 = u(s - v - w) \in (us, t^2\mathfrak{m})$, we have $u^q \in (s, t^2)^q$ and the same holds for v^q, w^q. Since $l_A(A/(s, t^2)^q) \sim 3q^2/4$, $l_A(\mathfrak{m}^{[q]}/J^{[q]})) \le 3q^2/4$, we have $e_{\mathrm{HK}}(A) \ge 9/4$ if $a \ge 2$.

Remark 5.8. (1) By the argument above, the only value between 2 and 9/4 which is $e_{\mathrm{HK}}(A)$ for some Cohen–Macaulay local ring of dimension 2 is 11/5.

(2) After submitting this paper, we showed that if $\dim A = 2$ and if equality holds in (5.5.1), then $G(\mathfrak{m})$ (the associated graded ring of A with respect to $\mathfrak{m}$) is isomorphic to the eth Veronese subring of $k[X, Y]$. Details will appear in [WY].

(3) After this paper was submitted, Seibert gave me an e-mail saying that he and Kunz also proved Theorem 2.15.

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